· (Wirtinger Inequality)

Let f be a 2π-periodic C^1 function with $\hat{f}(0)=0$.

Then $$\int_0^{2\pi}|f(x)|^2dx \le \int_0^{2\pi}|f'(x)|^2dx$$

The equality holds iff $f(x)=A\cos x+B\sin x$.

Pf: By Parseval Identity, we have

$$\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^2dx = \sum_{n=-\infty}^{\infty}|\hat{f}(n)|^2$$

$$\frac{1}{2\pi}\int_0^{2\pi}|f'(x)|^2dx = \sum_{n=-\infty}^{\infty}|\hat{f'}(n)|^2$$

When $n\neq 0$,

$$\hat{f'}(n) := \frac{1}{2\pi}\int_0^{2\pi} f'(x)e^{-inx}dx$$

$$= \frac{1}{2\pi}\left(f(x)e^{-inx}\Big|_0^{2\pi} - \int_0^{2\pi} f(x)(-ine^{-inx})dx\right)$$

$$= \frac{in}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx = in\hat{f}(n).$$

When $n=0$, since f is 2π-periodic and C^1

$$\hat{f'}(0) = \frac{1}{2\pi}\int_0^{2\pi} f'(x)\,dx = 0.$$

Then
$$\frac{1}{2\pi}\int_0^{2\pi} |f'(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\hat{f'}(n)|^2$$
$$= \sum_{n=-\infty}^{\infty} n^2 |\hat{f}(n)|^2$$
$$\geq \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$
$$= \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx.$$

The equality holds iff

$$\hat{f}(n) = 0, \quad \forall\ |n| > 1.$$

Then $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$ and $S_N f \rightrightarrows f$.

So
$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$$
$$= \hat{f}(1) e^{ix} + \hat{f}(-1) e^{-ix}$$
$$= \hat{f}(1)(\cos x + i \sin x) + \hat{f}(-1)(\cos x - i \sin x)$$

$$= A\cos x + B\sin x$$

where $\quad A = \hat{f}(1) + \hat{f}(-1)$

$\quad\quad\quad B = i(\hat{f}(1) - \hat{f}(-1))$.

$\square$

Rmk: • Let f be any 2π-periodic C^1 function.

Then $\displaystyle\int_0^{2\pi} |f(x) - \hat{f}(0)|^2 dx \le \int_0^{2\pi} |f'(x)|^2 dx$.

Pf: Let $h(x) = f(x) - \hat{f}(0)$.

Then $\hat{h}(0) = \hat{f}(0) - \hat{f}(0) = 0$.

$h'(x) = f'(x)$

Applying Wirtinger Inequality to h, we will get this.

• If f is T-periodic function with

$\hat{f}(0) := \dfrac{1}{T}\displaystyle\int_0^T f(x)\,dx = 0$, then

$$\int_0^T |f(x)|^2 dx \le \left(\frac{T}{2\pi}\right)^2 \int_0^T |f'(x)|^2 dx .$$

The equality holds iff

$$f(x) := A\cos(2\pi x/T) + B\sin(2\pi x/T).$$

- C^1 can be replaced by piecewise C^1.

- Let f and g be 2π-periodic C^1 function with $\hat{f}(0) = 0$.

Then $$\left|\int_0^{2\pi} f(x)\overline{g(x)}\,dx\right|^2 \le \int_0^{2\pi} |f(x)|^2 dx \int_0^{2\pi} |g(x)|^2 dx$$

Rmk: If we let $g = f$, then we get

$$\left(\int_0^{2\pi} |f(x)|^2 dx\right)^2 \le \int_0^{2\pi} |f(x)|^2 dx \int_0^{2\pi} |f'(x)|^2 dx$$

$$\Rightarrow \int_0^{2\pi} |f(x)|^2 dx \le \int_0^{2\pi} |f'(x)|^2 dx .$$

This is a generalization of Wirtinger Inequality.

Pf: $$\left| \int_0^{2\pi} f(x)\,\bar{g}(x)\,dx \right|^2$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \hat{f}(0) = 0$$

$$= \left| \int_0^{2\pi} f(x)\,\overline{(g(x) - \hat{g}(0))}\,dx \right|^2$$

$$\leq \int_0^{2\pi} |f(x)|^2 dx \int_0^{2\pi} |g(x) - \hat{g}(0)|^2 dx$$ Cauchy-Schwarz

$$\leq \int_0^{2\pi} |f(x)|^2 dx \int_0^{2\pi} |g'(x)|^2 dx$$ Bcuk of Wirtinger

$\square$

- Let f be a 2π-periodic C^1 function with $f(0)=0$.

Then $\int_0^{2\pi} |f(x)|^2 dx \le 4 \int_0^{2\pi} |f'(x)| dx$.

Pf: Let $g(x) = \begin{cases} f(x), & x \in [0, 2\pi] \\ -f(-x), & x \in [-2\pi, 0] \end{cases}$.

and extend g to be a 4π-periodic function.

Since $f(0)=0$, then g is well-defined.

Applying generalized Wirtinger Inequality to g.

$$\int_0^{4\pi} |g(x)|^2 dx \leq \left(\frac{4\pi}{2\pi}\right)^2 \int_0^{4\pi} |g'(x)|^2 dx$$

$$\parallel \qquad\qquad\qquad\qquad \parallel$$

$$\int_0^{4\pi} |f(x)|^2 dx \qquad\qquad 4\int_0^{4\pi} |f'(x)|^2 dx$$

$$\parallel \qquad\qquad\qquad\qquad \parallel$$

$$2\int_0^{2\pi} |f(x)|^2 dx \qquad\qquad 8\int_0^{2\pi} |f'(x)|^2 dx$$

$$\Rightarrow \quad \int_0^{2\pi} |f(x)|^2 dx \leq 4\int_0^{2\pi} |f'(x)| dx$$

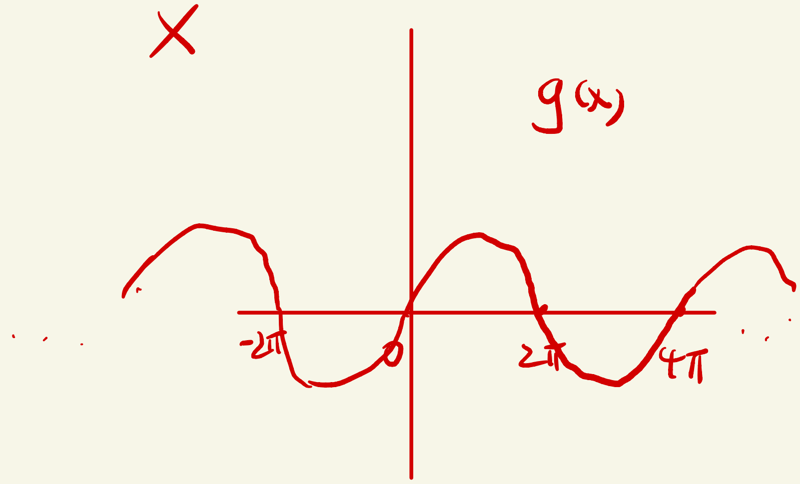
The equality holds iff

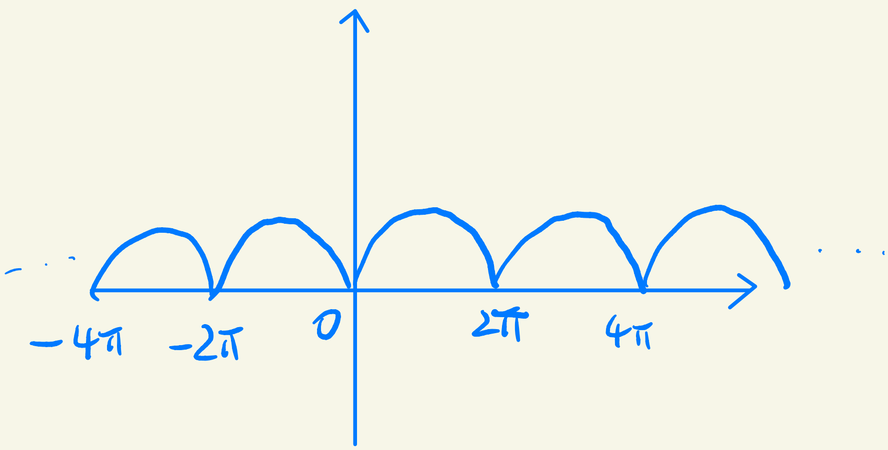
$$g(x) = A\cos\left(\frac{2\pi x}{4\pi}\right) + B\sin\left(\frac{2\pi x}{4\pi}\right)$$

$$= \cancel{A\cos\left(\frac{x}{2}\right)} + B\sin\left(\frac{x}{2}\right).$$ (since g is odd)

$$= B\sin\frac{x}{2}.$$

So $f(x) = B\sin\frac{x}{2}$ ×





$f(x) = B\sin\frac{x}{2}$, $x \in [0, 2\pi]$

and then extended to a 2π-periodic function

$\square$

Rmk: For 2π-periodic C^1 function, if

we $\hat{f}(0) = 0$ or $f(0) = 0$, we can both control the L^2-norm of f

by the L^2-norm of its derivative.

$\square$